Exam PPP

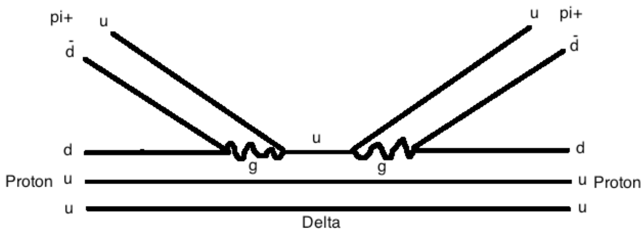
11 April 2017

- Put your name and student number on each answer sheet.
- Answer all questions short and to the point, but complete; write legible.
- Final point grade = total number of points/11 + 1

1. **Pion-nucleon scattering (20 points)**
Consider the scattering of a pion-beam on a proton target, $\pi^+ + p \to \pi^+ + p$. The pion-beam momentum is chosen such that the reaction takes predominantly place via the production and decay of an intermediate Δ resonance, e.g. $\pi^+ + p \to \Delta^{++} \to \ldots$. The Δ resonance has an isospin (I), spin (J) and parity (P) of $I(J^P)$=3/2(3/2$^+$) with a mass of M_Δ=1232 MeV. The isospin/spin/parity of the proton is $I(J^P)$=1/2(1/2)$^+$. The pion has a isospin/spin/parity of $I(J^P)$=1(0$^-$). with a mass of M_p=940 MeV (M_π=140 MeV), respectively.

   a) Historically, the experimental observation of the Δ resonance has led to the theory of Quantum Chromodynamics in which quarks carry a color degree of freedom. Explain how an observation of the Δ gives rise to the need of color assuming (iso)spin symmetry.
   The wavefunction of a three-identical-fermion system has to be antisymmetry with respect to swapping (relabeling) the various fermion in the system. This corresponds to the Pauli exclusion principle. In the case of isospin symmetry, the two u and d quarks can be considered as identical. Similarly, for the spin projection of these two quarks. The wavefunction of a baryon has a component representing the spin-configuration, a component representing the isospin, a component representing the coordinates of the quarks. For a spin-3/2 and isospin-3/2 baryon (such as the Delta), twe first two components are even in symmetry (corresponding to the "4"). The space-component is even as well, since the Delta has no angular momentum (factor -1^L due to spherical harmonic functions). With an additional odd component, Pauli exclusion principle is violated. This antisymmetry component stems from the color degree of freedom of quarks (to be precise the antisymmetric singlet component).

   b) Sketch the Feynman diagram at the quark level of the scattering process with an intermediate Δ. Note that the process takes place via the strong interaction.



   c) Calculate the pion momentum that would produce a Δ at its resonance mass.
   The trick is to evaluate the invariant $E^2_{\rm tot} - P^2_{\rm tot}$ in the laboratory and center-of-mass frame. In the center-of-mass frame, this expression is equal to M^2_Δ if the beam momentum is chosen such to be at the resonance mass. This factor is equal to the same evaluated in

the laboratory system, hence:

$$\begin{aligned}(E_\pi + M_p)^2 - P_\pi^2 &= M_\Delta^2; \\ \Rightarrow E_\pi &= \frac{M_\Delta^2 - M_\pi^2 - M_p^2}{2M_p} = 327 \text{ MeV}.\end{aligned}$$

Hence, the pion-beam momentum becomes $P_\pi = \sqrt{E_\pi^2 - M_\pi^2} = 295$ MeV/c.

d) Use isospin conservation to estimate the cross section ratio

$$\frac{\sigma(\pi^+ + p \to \pi^+ + p)}{\sigma(\pi^+ + n \to \pi^+ + n)},$$

assuming that both reactions take place at the same center-of-mass energy and via an intermediate Δ resonance. Motivate your answer.

Isospin invariance together with Δ as intermediate state imply that only the $I = 3/2$ component of the πN system in the initial- and final-state participate in the reaction. The $\pi^+ p$ system is always in an $I = 3/2$ state. The $\pi^+ n$ has a $I = 1/2$ and $I = 3/2$ component. From the Clebsch-Gordon table $(1 \times 1/2)$, we can read of the fraction of $I = 1/2$ and $I = 3/2$. The fraction of $I = 3/2$ is $\sqrt{1/3}$. The amplitude ($=< \pi^+ n | H_{3/2} | \pi^+ n >$) of the $\pi^+ n$ channel is, therefore, a factor 3 smaller than the $\pi^+ p$ case. The cross section (amplitude squared) ratio is, therefore, about 9.

2. **Charmonium and tetraquarks (20 points)**

The mysterious $X(3872)$ is a particle with a hidden charmonium content (e.g. it contains $c\bar{c}$ pair). It has a mass of 3872 MeV and a spin-(charge) parity of J^{PC}=1^{++}. It was discovered by Belle in 2003, and, more recently, observed by BESIII in the reaction $e^{+}+e^{-} \rightarrow \psi(4260) \rightarrow \gamma+X(3872)$, where $\psi(4260)$ is a vector charmonium state (J^{PC}=1^{--}) with a mass of 4260 MeV and γ a radiative photon (J^{PC}=1^{--}).

a) It is extremely rare to produce the $X(3872)$ in the electromagnetic annihilation process $e^{+}e^{-} \rightarrow X(3872)$. Explain why such a direct production is practically impossible. Motivate your answer.

The electron-positron annihilation process takes place via an intermediate virtual photon. Since the photon has a spin-parity of 1^{-}, it cannot couple directly to a 1^{+} state such as the $X(3872)$ in order to conserve parity in an electromagnetic interaction. Only via exotic diagrams, such as two-photon exchange this would be possible. However, that will contain more α factors, therefore, very suppressed.

b) Most of the $\psi(4260)$ states will decay into a pair of open-charm mesons, $D\bar{D}$. What is the reason on the quark level why such a decay is favorable? Motivate your answer. The mass of a D meson is 1870 MeV and it is composed of a (anti) charm quark together with a (anti) light u or d quark.

Charmonium states above the open-charm threshold, e.g. their mass larger than the sum of the two D-mesons, can decay into a $D\bar{D}$ pair. This process is OZI allowed and it involves only one gluon exchange that couples to a quark and an antiquark (u or d). Other possible decay modes of a $c\bar{c}$ are suppressed with respect to the open-charm decay, since they involve either photons (e.m. interactions are suppressed because of α factors) or more than one gluon exchange, which is suppressed since α_S is about 0.2 in this mass regime. Weak decays are suppressed because of the massive Z boson.

c) The $X(3872)$ cannot decay into two open-charm mesons, $D\bar{D}$ ($J^{P}(D)$=0^{-}), which makes its width very small. Argue why this decay is forbidden in the strong interaction.

For the decay of a 1^{+} state into two 0^{-} particles, the relative orbital angular momentum between the two open-charm mesons has to be one in order to conserve total angular momentum. This, however, is not possible in a strong decay since parity will be violated. The parity in the final-state is namely $P = P(D) \times P(\bar{D}) \times (-1)^{L}$, which is odd for $L = 1$.

d) One of the strongest decay modes of the $X(3872)$ turns out to be $X(3872) \rightarrow J/\psi+\rho$, where J/ψ is the lightest vector charmonium state ($c\bar{c}$) and ρ is a meson composed of light (u, d) quarks with an isospin of one. Explain why such an observation favors the interpretation of the $X(3872)$ as a tetraquark state in stead of a conventional $c\bar{c}$ charmonium state. The tetraquark in this case has a quark contents of $q\bar{q}c\bar{c}$ whereby q represents one of the light quarks u or d.

If the $X(3872)$ would be a conventional charmonium state, its isospin would be zero. Hence, the decay into a J/ψ (isospin zero) and ρ (isospin one) would be suppressed since isospin is not conserved in such a reaction. In the case, the $X(3872)$ is a tetraquark including two light u or d quarks, it can have an isospin of either zero or one. The latter option would allow this reaction to take place, conserving isospin.

3. **The pion and sigma (20 points)**

The lightest hadron in nature is the pion, π, with J^P=0^- and a mass of 140 MeV. This meson is also known as the (pseudo) Goldstone particle in hadron physics. The chiral-partner state of the pion is the sigma, σ, with J^P=0^+ and a significantly larger mass of about 600 MeV. It is believed that the mass of hadrons originate from a spontaneous chiral-symmetry breaking. On the quark level, the π and σ mesons are composed of $q\bar{q}$ where q=u or d.

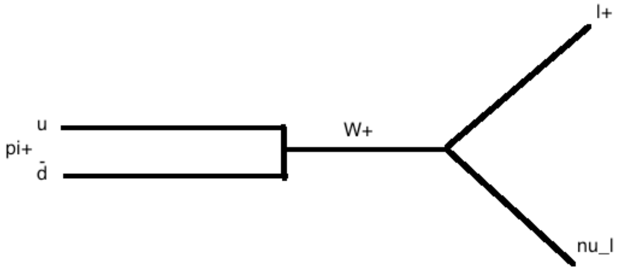
a) What is the origin of chiral symmetry and explain qualitatively the phenomena of spontaneous chiral-symmetry breaking in the context of the pion and sigma fields.

The origin of the chiral symmetry originates from the approximation of massless quarks. In that case, the Lagrangian of QCD can be formulated such that left-handed and right-handed chiral components of the wavefunctions are decoupled (note that mass coupled left and right-handed fields). Moreover, the Lagrangian has a symmetry with respect to "rotating" left-handed fields to right-handed fields and vica versa. One believes that the underlying potential is such that the ground state of this system is spontaneously broken. Although the potential is still symmetric with respect to the chiral rotation, the ground state is not. The picture would be a mexican hat like potential. The lowest lying state from such a potential is the Goldstone boson: a field moving in the minimum track of the hat, hence without any mass (potential is flat). This corresponds to the pion. The field in radial direction can be labeled as the σ field, which quanta has a mass because of this symmetry breaking phenomena (similarity to Higgs field).

b) The q and $\bar{q}$ are both spin-1/2 fermions. Analyze the possible sum of the spins and orbital angular momenta between the two quarks in the case of a π and in the case of a σ meson. Motivate your answer.

The pion has a $J^P = 0^-$. A negative parity can be obtained when the relative orbital angular momentum (L) between the q and $\bar{q}$ is even, since $P = P(q)P(\bar{q})(-1)^L$ whereby $P(q) = -P(\bar{q})$. The two spins can couple to $S = 0$ or $S = 1$. Since the total spin J is zero, only the combination of $L = 0$ and $S = 0$ is possible. The sigma has a $J^P = 0^+$. The positive parity can be obtained when L is odd. To combine $S = 0, 1$ with L odd, such that $J = 0$, only the possibility of $L = 1$ and $S = 1$ remains for the σ.

c) Electrically charged pions decay via the weak interaction into a (anti) neutrino and a charged lepton. Sketch the Feynman diagram of such a decay.



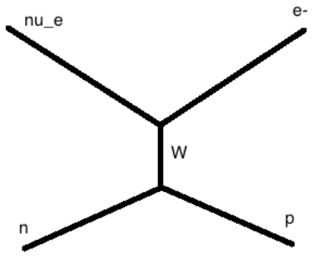
d) Explain why the weak decay rate for the pion into a muon+neutrino is much larger than that the decay rate into an electron/positron+neutrino final state. The mass of a pion is 140 MeV, the mass of an electron/positron is 0.5 MeV, and the mass of a muon is 105 MeV. The neutrino mass can be neglected.

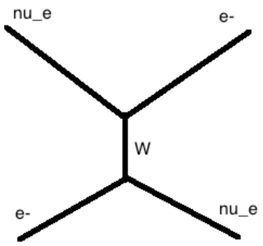
Take the example of a negatively charged pion decaying in its rest frame: $\pi^- \rightarrow \ell^- + \bar{\nu}_\ell$ (similar conclusion can be drawn from a π^+ decay). The antineutrino and charged lepton will move in opposite directions. The antineutrino has a spin of 1/2 and practically massless. According to the V-A theory of weak interactions, it is always right-handed in chirality and in helicity. This means that the spin points in the same direction as its momentum. In order to give a total spin of zero (=spin of the initial pion), the spin direction of the ℓ^- is also in the same direction as its momentum. This means that the helicity of the ℓ^- is right-handed. For a particle with an energy that is much larger than its mass, helicity is close to chirality. For the electron case, the mass is very small and hence it is necessary that its chirality is mostly right-handed. This is not allowed in the V-A weak interaction. On the other hand, the muon will still have a large left-handed chirality component since its helicity is not close to its chirality because it has a relatively large mass with respect to its energy. Hence the muon decay will be much stronger.

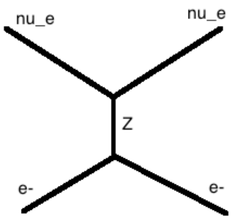
4. **Solar neutrinos (20 points)**

Nuclear (fusion) processes that take place in the Sun provide a source of electron neutrinos, ν_e, with a large variation in energies that can go up to at most 20 MeV. The detection of electron neutrinos can take place via Cherenkov radiation of charged leptons in purified water such as the Super-Kamiokande setup in Japan. The (dis)appearance probability of $\nu_e \rightarrow \nu_\mu$ can be approximated (assuming oscillations among two neutrino flavors) by $P_{e\mu} = \sin^2(2\theta_{12})\sin^2(1.27\Delta m^2\, L/E)$ with θ_{12} the corresponding mixing angle, Δm^2 the mass-square difference between the two neutrino mass eigenstates in units of eV^2, L the distance from the neutrino source in units of kilometers, and E the neutrino energy in units of GeV. The diameter of the Sun is about 1.4×10^6 kilometers. The mass-square difference, Δm^2 is about 8×10^{-5} eV^2.

a) Sketch the dominant weak processes (neutral and charged currents) that can take place between the electron-neutrino and nucleons or atomic electrons that produce fast-moving electrons in the final state. Why is the detection sensitivity for electron-neutrinos larger than for neutrinos with another flavor?







ν_μ and ν_τ will participate in the NC with collisions with atomic electrons. Electron neutrinos also participate in CC giving rise to fast moving electrons in the final-state. One might argue that ν_μ and ν_τ can interact via CC giving fast moving electrically charged muons and tauons in the final state. These have a very high threshold, much larger than 20 MeV.

b) An electron-neutrino survival probability, P_{ee}, of 0.58 has been measured for low energy electron neutrinos; See Fig. 2. Assume that the MSW effect does not play a role in this case (E_ν<2 MeV). Estimate the angle θ_{12} from such an observation. Motivate your answer.
The survival probability $P_{ee} = 1 - P_{e\mu}$ (since neutrinos do not decay in flight). Since neutrinos in the Sun are produced with a variety of energies and at various places, the factor $\sin^2(1.27\Delta m^2\, L/E)$ will average to 1/2. Also note that the oscillation wavelengths are significantly smaller than the size of the Sun. This yields to $\theta_{12} = 33°$.

c) The MSW effect will play a role in the neutrino oscillation process for neutrino energies larger than about 2 MeV. The survival probability will significantly be reduced compared to the vacuum conditions as can be observed from Fig. 2. Describe qualitatively the MSW effect and its origin.
The MSW effect describes oscillations in matter. All neutrino flavors interact in matter via NC, however, ν_e is the only one that also interacts via CC. The consequence of this is that the potential seen by the electron neutrino is fastly different than that of the other

neutrino flavors. This modifies the interaction Hamiltonian in such a way that the flavor eigenstate ν_e is about the same as the mass eigenstate ν_2 in matter. When the neutrinos propagate to the Earth it will mostly be in a ν_2 state. The consequence is that the survival probability, P_{ee}, will be reduced to $\sin^2\theta_{12}$ which is about 0.3.

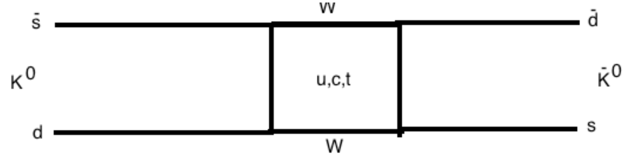
d) Neutrino oscillations are only sensitive to the mass-square difference between the neutrino eigenstates. Describe briefly an experimental method that will be able to set an absolute mass scale on the neutrino mass.

Possible answers are: 1) Measure the endpoint of the beta-decay spectrum. The maximum energy of a beta-electron depends on the Q-value, and, hence, on the mass of the antineutrino. Example: tritium decay (KATRIN experiment in Karlsruhe which aims to measure down to 0.2 eV). 2) Cosmology. the fluctuations in the cosmic microwave background spectrum are sensitive to the earlier mass-energy balance in the Universe and are related to the large-scale structure formations. Neutrino masses play a role in the corresponding model. At present there is an upper limit of 600 meV corresponding to the sum of the three mass eigenstate. 3) neutrinoless double beta decay: A process that is very rare, since it involves two W exchanges and a helicity flip by the virtual neutrino, but it might occur when the neutrino is a Majorana particle and when the mass of the neutrino is finite. At present the upper limit is a few hundreds of meV.

5. **Kaons (20 points)**
To study the $K^0-\bar{K}^0$ oscillations, a pure beam of K^0 ($\bar{s}d$) mesons is prepared using the reaction $\pi^-p \to K^0\Lambda$ slightly above its production threshold. The Λ hyperon is the lightest baryon with a strange quark (uds). Kaons have a spin-parity of J^P=0^-.

a) Why are there no $\bar{K}^0$ mesons produced during these π^-p collisions.
In order to conserve baryon number, strangeness, etc., the "first" possible production of $\bar{K}^0$ occurs in the process $\pi^-p \to K^0\bar{K}^0n$. The threshold of this reaction is larger than the one considered, and, hence, it will not take place.

b) Sketch the underlying (quark-level) process that can cause $K^0-\bar{K}^0$ oscillations.



c) The K_S and K_L are the mass eigenstates with lifetimes of 90 ps and 50 ns, respectively. In the case of CP invariance, the K_S and K_L are also CP eigenstates. Construct the CP-even (K_S) and CP-odd (K_L) wavefunctions in terms of the K^0 and $\bar{K}^0$ components.

$$\begin{aligned}|K_S> &= \mathrm{CP-even} = (|K^0> -|\bar{K}^0>)/\sqrt{2};\\ |K_L> &= \mathrm{CP-odd} = (|K^0> +|\bar{K}^0>)/\sqrt{2},\end{aligned}$$

whereby we define $CP|K^0>=-|\bar{K}^0$ and $CP|\bar{K}^0>=-|K^0>$.

d) The K_S decays predominantly (close to 100%) into two pions ($J^P(\pi)$=0^-). The $K_L \to \pi\pi$ process has a tiny branching fraction of $2{\times}10^{-3}$. Explain this observation by evaluating the CP eigenvalue of the two-pion system.
The two-pion system is either $\pi^0\pi^0$ (1) or $\pi^+\pi^-$ (2). (1) In order to conserve total angular momentum, the two π^0 are in a relative orbital angular momentum of $L=0$. The parity is, therefore, even. The charge parity for this system is the multiplication of the charge parity of each π^0, which is even. CP is, therefore, even in the $\pi^0\pi^0$ case. Also note that parity is violated in this reaction. Also note that for two identical bosons in the final state, the corresponding wave function has to be symmetric with respect to exchanging particle labels. With L even this is the case. (2) Similar as (1), $L=0$ and, therefore, parity is even. The charge parity operator has the same effect as the parity operator. Hence the eigenvalues of the charge parity is $(-1)^L$, and, therefore also even. CP is, therefore, even in the $\pi^+\pi^-$ case. Since K_L is CP-odd, it will hardly decay into a CP-even state. CP is only slightly broken in the weak interaction.

e) Describe a method to count the intensity of K^0 and $\bar{K}^0$ mesons as function of distance from the production target. Estimate the fraction of K^0 and $\bar{K}^0$ after a few meters from target. Motivate your answers.
The fraction of K^0 and $\bar{K}^0$ can be obtained by observing their semileptonic decay. The charge sign of the final-state lepton is directly related to a K^0 or $\bar{K}^0$, since $K^0 \to \pi^-\ell^+\nu_\ell$ and $\bar{K}^0 \to \pi^+\ell^-\bar{\nu}_\ell$. After a few meters, most of the K_S will have decayed since its $c\tau$ is only 2.7 cm. The remaining beam is, therefore, a K_L. According to c), the fraction of K^0 and $\bar{K}^0$ will be approximately equal for a pure K_L beam.

This exam has been drafted by J.G. Messchendorp and verified by C.J.G. Onderwater.

43. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	…
		M	M	…
m₁	m₂	Coefficients		
m₁	m₂			
⋮	⋮			

1/2×1/2

		1
		+1
+1/2	+1/2	1

		1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1
		−1
−1/2	−1/2	1

1×1/2

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

2×1

		3
		+3
+2	+1	1

		3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

		3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

		3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

		3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

		3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

		3
		−3
−2	−1	1

1×1

		2
		+2
+1	+1	1

		2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2
		−2
−1	−1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\sin^2\theta\, e^{2i\phi}$$

2×1/2

		5/2
		+5/2
+2	+1/2	1

		5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

		5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

		5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

		5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

		5/2
		−5/2
−2	−1/2	1

3/2×1/2

		2
		+2
+3/2	+1/2	1

		2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

		2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

		2
		−2
−3/2	−1/2	1

3/2×1

		5/2
		+5/2
+3/2	+1	1

		5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

		5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

		5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

		5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

		5/2
		−5/2
−3/2	−1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\,Y_\ell^m\, e^{-im\phi}$$

$$\boxed{\begin{aligned}&\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle\\ &= (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle\end{aligned}}$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

3/2×3/2

		3
		+3
+3/2	+3/2	1

		3	2
		+2	+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

		3	2	1
		+1	+1	+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

		3	2	1	0
		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

		3	2	1
		−1	−1	−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

		3	2
		−2	−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

		3
		−3
−3/2	−3/2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1}_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

		7/2
		+7/2
+2	+3/2	1

		7/2	5/2
		+5/2	+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

		7/2	5/2	3/2
		+3/2	+3/2	+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

		7/2	5/2	3/2	1/2
		+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

		7/2	5/2	3/2	1/2
		−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

		7/2	5/2	3/2
		−3/2	−3/2	−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

		7/2	5/2
		−5/2	−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

		7/2
		−7/2
−2	−3/2	1

2×2

		4
		+4
+2	+2	1

		4	3
		+3	+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

		4	3	2
		+2	+2	+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

		4	3	2	1
		+1	+1	+1	+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

		4	3	2	1	0
		0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

		4	3	2	1
		−1	−1	−1	−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

		4	3	2
		−2	−2	−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

		4	3
		−3	−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

		4
		−4
−2	−2	1

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}\,(2\cos\theta-1)$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\,\sin\theta\,\cos\theta$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2 \qquad d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}\,(2\cos\theta+1) \qquad d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 1: Clebsch-Gordan coefficients.

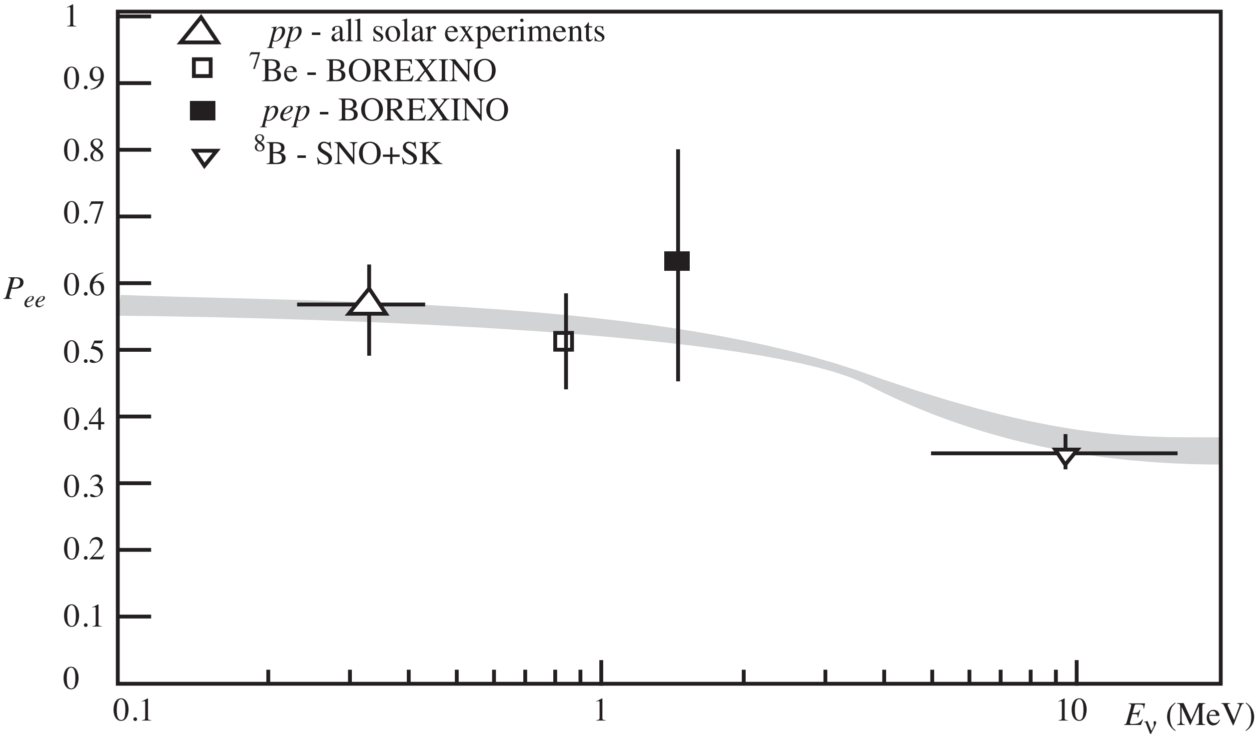


Figure 2: Electron neutrino survival probability at different components of the solar energy spectrum.